

# Coupled Integral Equations for the Nucleon and Pion Electromagnetic Form Factors\*

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The dispersion relations for the nucleon isotopic vector form factors and the pion form factor which take into account contributions from both the  $2\pi$  and  $NN$  intermediate states become a set of coupled integral equations for the form factors if the four amplitudes  $(\pi\pi|NN)$   $(\pi\pi|\pi\pi)$   $(NN|\pi\pi)$   $(NN|NN)$  are assumed known. If these four amplitudes are replaced by their Born approximation values and spin and certain kinematic factors are neglected, the resulting set of coupled singular integral equations can be solved exactly. Comparison of these exact solutions with the form factors obtained from the usual approximation of retaining only the lowest mass state (i.e., the  $2\pi$  state) confirms the hope that high-mass states do not contribute much to dispersion integrals. It is also of interest that these solutions are obtained from dispersion relations without subtractions and satisfy the necessary conditions that they vanish at infinite momentum transfer and take on the value  $e$  at the origin for all values of the coupling parameters appearing in the equations.

## I. INTRODUCTION

IN this paper we ask and attempt to answer two somewhat unrelated questions having to do, basically, with the nucleon electromagnetic structure problem. There has been a considerable amount of effort expended on this problem in the last few years using dispersion relations for the form factors as a means of attack.<sup>1</sup> These approaches have all followed more or less the same line—namely, writing the form factors as integrals over their imaginary parts, expressing these imaginary parts as sums of contributions from many intermediate states, and attempting to calculate the contributions of the simplest such states with the hope that no others are important. As shown schematically in Fig. 1, the imaginary part of the nucleon isotopic vector form factor receives contributions from intermediate states of two pions, and of two nucleons, among others. These contributions may be calculated explicitly in terms of other processes; for example, the contribution of the two-pion state involves the pion electromagnetic form factor (i.e., the amplitude for the reaction  $\gamma \rightarrow \pi + \pi$ ) and the amplitude for the process  $\pi + \pi \rightarrow N + \bar{N}$ .

To evaluate the pion form factor, a similar procedure may be followed. Figure 2 indicates the two-pion and two-nucleon intermediate state contributions to the imaginary part of the pion form factor. It is clear from Figs. 1 and 2 that if all intermediate states were retained in computing the imaginary parts of the nucleon and pion form factors, we would have a coupled problem. Even ignoring all intermediate states other than those shown explicitly in Figs. 1 and 2 will still leave us with coupled dispersion relations expressing the pion form factor as an integral over the nucleon form factor, among other things, and with the nucleon form factor

expressed as an integral over the pion form factor, plus other terms.

This coupled problem has in previous discussions been uncoupled by the following argument. Consider the two-nucleon contribution to the pion form factor. This is an integral of which the integrand is proportional to the product of the nucleon form factor times the amplitude that a nucleon antinucleon pair annihilates into two pions in the  $p$  state. This annihilation amplitude is bounded by the limitations imposed by unitarity. Hence if the nucleon form factor doesn't get too large, it is possible to bound the two-nucleon contribution for small values of the argument of the pion form factor. Since this bound is small, the two-nucleon state is neglected.

Now there are two things wrong with this argument; the first is that it is not known that the nucleon form factor can't get too large. After all, the nucleon form factor depends on the pion form factor, and the value of the pion form factor depends on how big the two-nucleon contribution to it is, and so on. The second thing wrong with the usual argument is that even assuming the nucleon form factor stays small, a bound is provided only for small momentum transfers in the pion form factor. However, in evaluating the nucleon form factor, an integral over the pion form factor is needed. Thus we need the pion form factor for values of momentum transfer where unitarity poses no restriction. There is, therefore, no reason to omit the two-

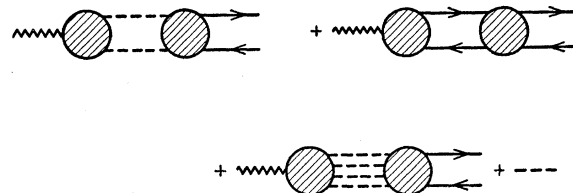


FIG. 1. Intermediate states contributing to the nucleon isotopic vector form factors. Dotted lines denote pions, heavy lines nucleons, and the wavy line is a photon.

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<sup>1</sup> P. Federbush, M. L. Goldberger, and S. B. Treiman, Phys. Rev. **112**, 642 (1958); G. F. Chew, R. Karplus, S. Gasiorowicz, and F. Zachariasen, Phys. Rev. **110**, 265 (1958).

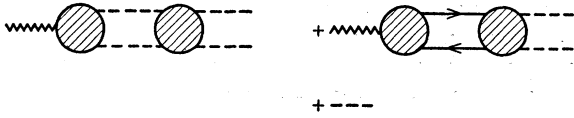


FIG. 2. Intermediate states contributing to the pion form factor.

nucleon contribution and the nucleon form factor is as a result also needed for high momentum transfer. Hence we can no longer use the unitarity argument to say that the two-nucleon state contribution to the nucleon form factor is negligible. But then there is no reason for neglecting the two-nucleon state contribution to either form factor and we are left with two coupled equations. It seems, then, that the effect of the coupling of the two form factors is a thing worth studying further. It may be that the usual assumption that the equations can be uncoupled by neglecting the two-nucleon state in the pion form factor is in fact valid, but the usual argument is not sufficient to show that it is valid.

The first problem we set ourselves, therefore, is to construct a pair of equations sufficiently similar to the actual coupled dispersion relations so that the solutions will have the same qualitative behavior as the actual ones, but also sufficiently different so that they may be solved exactly. We thus hope to be able to see whether the uncoupling of the equations is or is not a reasonable approximation.

Once one is thinking in terms of coupled dispersion relations of this type, a second quite different question suggests itself. It has usually been the practice to use subtracted dispersion relations for charge form factors, since the subtraction constant is the electromagnetic charge of the particle which is a known number. If, however, unsubtracted dispersion relations are used, the charge may be written as an integral over the imaginary part of the form factor.<sup>2</sup> Furthermore, the kernels in these coupled equations will depend on the meson-nucleon coupling constant. Thus, depending on the uniqueness of the solutions of these (homogeneous) equations, and depending on whether or not the equations turn out to be eigenvalue equations, one might hope to obtain some restrictions on the possible values of the electron charge and on the meson-nucleon coupling constant. We shall indicate later in more detail how such restrictions might arise. For the moment, it is sufficient to say that it is obviously of interest to ask about the existence and uniqueness of solutions to such coupled homogeneous singular integral equations.

What we have actually done, then, is to invent equations which are a modification of the simplest nontrivial coupling of pion and nucleon electromagnetic form factors, namely, those shown diagrammatically in Figs. 1 and 2, assuming all particles have spin zero. We have solved these equations exactly, and therefore

answered the above questions for these somewhat fictitious equations.

The answers to the second set of questions, about the unsubtracted equations, are first that solutions do indeed exist, so that the use of unsubtracted dispersion relations seems justified; second, that there is enough nonuniqueness in the solution so that the charge is not determined. Finally, the equations are not eigenvalue equations, and solutions exist for any value of the meson-nucleon coupling constant.

## II. THE MODEL

Our purpose is first of all to understand the mathematical properties of coupled dispersion relations, and we shall therefore begin by eliminating all complications not relevant to this purpose. We suppose that protons, mesons, and photons all have spin zero, and forget about neutrons and neutral mesons. With these simplifications, the proton electromagnetic structure problem can be set up as follows. The proton form factor  $F_2$  is defined by

$$\langle p_1 p_2^{(-)} | j_\gamma(0) | 0 \rangle = \frac{1}{(4E_{p1}E_{p2})^{\frac{1}{2}}} F_2((p_1 + p_2)^2), \quad (1)$$

where  $p_1, p_2$  denote the 4-momenta of a pair of protons produced by a photon.  $j_\gamma(x)$  is the Heisenberg photon current operator, defined in terms of the photon field operator  $A$  by

$$\square A(x) = j_\gamma(x). \quad (2)$$

$E_{p1}, E_{p2}$  are the time components of  $p_1$  and  $p_2$ , and finally  $\langle p_1 p_2^{(-)} |$  is the incoming wave Heisenberg state of the proton pair.

The proton form factor will satisfy a dispersion relation of the form

$$F_2(p^2) = -\frac{1}{\pi} \int \frac{\text{Im} F_2(p'^2)}{p'^2 - p^2 - i\epsilon} dp'^2. \quad (3)$$

As has already been discussed in the introduction, we assume the dispersion relation to hold in the unsubtracted form. This means that the electromagnetic charge of the proton is given by

$$F_2(0) = e = -\frac{1}{\pi} \int \frac{\text{Im} F_2(p'^2)}{p'^2} dp'^2. \quad (4)$$

Whether this equation is actually an identity in  $e$ , or turns out to be a method by which  $e$  can be calculated is one of the questions we wish to try to answer.

The imaginary part of  $F_2$  is as usual given by<sup>1</sup>

$$\text{Im} F_2(p'^2) = -(E_{p1}/2)^{\frac{1}{2}} \sum_n (2\pi)^4 \delta^4(P_n - p') \times \langle p_1 | j_p(0) | n \rangle \langle n | j_\gamma(0) | 0 \rangle, \quad (5)$$

where  $j_p$  is the proton current; that is, if  $\psi$  is the proton field,

$$j_p(x) = (\square + m_p^2)\psi(x). \quad (6)$$

<sup>2</sup> G. F. Chew (unpublished), S. D. Drell and F. Zachariasen, Phys. Rev. **111**, 1727 (1958).

The sum in Eq. (5) is supposed to run over all states which the conservation laws allow a photon to reach. In particular, a pair of mesons is one such state, and a pair of protons is another, and we shall write the contribution of these two states explicitly. Thus

$$\text{Im}F_2(p'^2) = K_{21}(p'^2)F_1^*(p'^2) + \sin\delta_2(p'^2) \exp[i\delta_2(p'^2)]F_2^*(p'^2) + R_2(p'^2). \quad (7)$$

Here  $\delta_2$  is the  $S$ -wave proton-antiproton scattering phase shift, to be evaluated at a total center-of-mass energy given by  $p'^2$ .  $F_1(p'^2)$  is the pion electromagnetic form factor, defined by

$$\langle q_1 q_2^{(-)} | j_\gamma(0) | 0 \rangle = \frac{1}{(4\omega_{q_1}\omega_{q_2})^{\frac{1}{2}}} F_1((q_1+q_2)^2), \quad (8)$$

where  $\omega_{q_1}$  and  $\omega_{q_2}$  are the energies of the two pions.  $K_{21}(p'^2)$  is the  $S$ -wave amplitude for the reaction  $\pi + \pi \rightarrow P + \bar{P}$ ; explicitly

$$K_{21}(p'^2) = -\left(\frac{E_{p_1}}{2}\right)^{\frac{1}{2}} \int \frac{d^3 q_1}{(2\pi)^3} \int \frac{d^3 q_2}{(2\pi)^3} (2\pi)^4 \delta^4(q_1+q_2-p') \\ \times \langle p_1 | j_p(0) | q_1 q_2^{(+)} \rangle \frac{1}{(4\omega_{q_1}\omega_{q_2})^{\frac{1}{2}}}. \quad (9)$$

Finally  $R_2(p'^2)$  is the contribution, in Eq. (5), of all the remaining states  $n$ .

An exactly analogous procedure can, of course, be followed for the pion electromagnetic form factor; it also is assumed to satisfy an unsubtracted dispersion relation, and its imaginary part may be written, in analogy to Eq. (7), as

$$\text{Im}F_1 = K_{12}F_2^* + \sin\delta_1 e^{i\delta_1} F_1^* + R_1. \quad (10)$$

Now it is easily shown that

$$\left(\frac{p'^2}{p'^2 - 4m_2^2}\right)^{\frac{1}{2}} K_{12}(p'^2) \equiv g(p'^2) \\ = \left(\frac{p'^2}{p'^2 - 4m_1^2}\right)^{\frac{1}{2}} K_{21}(p'^2), \quad (11)$$

where  $m_2$  is the proton mass and  $m_1$  the meson mass.

Thus we may write (using the fact that  $\text{Im}F_2$  is of course real)

$$\text{Im}F_2 = \frac{\text{Re}(\sin\delta_2 e^{i\delta_2})}{1 - \text{Im}(\sin\delta_2 e^{i\delta_2})} \text{Re}F_2 \\ + \left(\frac{p'^2 - 4m_2^2}{p'^2}\right)^{\frac{1}{2}} \frac{\text{Re}(gF_1^*)}{1 - \text{Im}(\sin\delta_2 e^{i\delta_2})} \\ + \frac{\text{Re}R_2}{1 - \text{Im}(\sin\delta_2 e^{i\delta_2})}, \quad (12)$$

and similarly for  $\text{Im}F_1$  with the indices 1 and 2 interchanged.

Up to this point we have made no approximations. One could, in principle, hope to continue by adding dispersion relations for the remaining unknown quantities, in particular for the remaining terms from  $\sum_n$  in Eq. (5) which are encompassed in  $R_1$  and  $R_2$ . Then one could, again in principle, and again hopefully, solve this large system of coupled dispersion relations. Such a program is, of course, impossible practically. We shall therefore drop the terms  $R_1$  and  $R_2$ , with no justification except to say that we are interested in solving the minimal nontrivial coupled dispersion relation problem, and that we hope this will have some relation to reality. At the very least, we should be able, in this approximation, to say something about how valid it is to uncouple the equations completely, as has been done in all previous attacks on the nucleon form factor.<sup>1</sup>

The set of equations obtained by setting  $R_1$  and  $R_2$  equal to zero in the expressions (12) for  $\text{Im}F_1$  and  $\text{Im}F_2$ , and then substituting these into the dispersion relations, is, however, still not a closed system of equations. It is still necessary to say something about the phase shifts  $\delta_1$  and  $\delta_2$  and the annihilation amplitude  $g$ . We could, of course, write dispersion relations for these, but this again violates our restriction that we wish to consider only the *simplest* nontrivial coupled problem. Alternatively we could use experiment for these quantities, but our purpose is to discuss the mathematical properties of the dispersion relations considered as a theory from which all things should follow. Therefore, in the same spirit by which we threw away  $R_1$  and  $R_2$ , we replace  $\delta_1$ ,  $\delta_2$  and  $g$  by Born approximation. In Born approximation, notice,  $\delta_1$ ,  $\delta_2$  and  $g$  are all real.<sup>3</sup>

Thus we end up with the coupled equations

$$F_1(p'^2) = \frac{1}{\pi} \int_{4m_1^2}^{\infty} \frac{\tan\delta_1(p'^2) \text{Re}F_1(p'^2)}{p'^2 - p^2 - i\epsilon} dp^2 \\ + \frac{1}{\pi} \int_{4m_2^2}^{\infty} \left(\frac{p'^2 - 4m_2^2}{p'^2}\right)^{\frac{1}{2}} \frac{g(p'^2)}{\cos^2\delta_1(p'^2)} \frac{\text{Re}F_2(p'^2)}{p'^2 - p^2 - i\epsilon} dp^2, \quad (13)$$

and similarly for  $F_2$  with the indices 1 and 2 interchanged. Here, remember,  $\delta_1$ ,  $\delta_2$  and  $g$  are to be taken from Born approximation.

These equations are now a closed system, and could in principle be solved. Because of the mathematical difficulty of handling such equations, however, one further simplification will be made, which is the following. The kernels (apart from the singular energy denominators from the dispersion integrals) all have the property of vanishing at the threshold of integration, and becoming constants at infinity. We shall replace

<sup>3</sup> Notice that the use of Born approximation in Eq. (12) cannot constitute a violation of unitarity, although its use in Eq. (7) could. For example, if we neglect  $g$  and  $R_1$ , we get  $\text{Im}F_1 = F_1^* \times \sin\delta_1 e^{i\delta_1}$  from Eq. (7), while from Eq. (12) we have (for real  $\delta$ )  $\text{Im}F_1 = \text{Re}F_1 \tan\delta_1$ . Replacement of  $\sin\delta_1 e^{i\delta_1}$  by Born approximation thus may violate the condition  $|\sin\delta_1 e^{i\delta_1}| < 1$ , but unitarity puts no bound on  $\tan\delta_1$  which is what occurs in our equation.

all these kernels by constants. This approximation will introduce singularities in the solutions at the threshold  $4m_1^2$  and  $4m_2^2$ , which were not present in the original equations, because the kernels (including the denominators) are no longer integrable at the thresholds. However, we feel that it will not alter the qualitative character of the solutions, or any important properties (such as, for example, whether solutions exist, or are unique) of the equations. To support this view, it may be worth observing that the equations may be solved in the equal mass case (pion mass equals nucleon mass) without neglecting the factor  $(p'^2 - 4m^2/p'^2)^{1/2}$ . Comparing the solution with this factor and without it shows that aside from the above mentioned singularity at  $p^2 = 4m^2$ , there is no difference in the qualitative behavior of the two cases.

The final set of equations on which we shall concentrate, then, is:

$$\begin{aligned} F_1(x) &= \frac{\lambda_{11}}{\pi} \int_{x_1}^{\infty} \frac{F_1(x')}{x' - x} dx' + \frac{\lambda_{12}}{\pi} \int_{x_2}^{\infty} \frac{F_2(x')}{x' - x} dx', \\ F_2(x) &= \frac{\lambda_{21}}{\pi} \int_{x_1}^{\infty} \frac{F_1(x')}{x' - x} dx' + \frac{\lambda_{22}}{\pi} \int_{x_2}^{\infty} \frac{F_2(x')}{x' - x} dx', \end{aligned} \quad (14)$$

together with the condition  $F_1(0) = F_2(0) = 1$ , where we removed the factor  $e$  from the definition of our form factors.

Here we have written the equations for the real parts of the form factors, and written  $F_1(x)$  in place of  $\text{Re}F_1(p^2)$ , etc. Thus principal values are to be understood on the integrals. The  $\lambda_{ij}$  are real numbers, and  $\lambda_{12} = \lambda_{21}$ . We wish, then, to study these equations and their solutions as functions of the  $\lambda$ 's and the masses  $x_1$  and  $x_2$ . In particular, we want to know (i) whether solutions exist; (ii) if they exist, do they exist for all values of  $\lambda_{ij}$ , or is it an eigenvalue problem (note that  $\lambda_{12} = \lambda_{21}$  is proportional to  $g^2$ , the square of the pion nucleon coupling constant. Thus if we have an eigenvalue problem, we could restrict the allowed values of  $g^2$ ). (iii) are the solutions unique; (iv) are there any values of  $\lambda_{ij}$ ,  $x_1$  and  $x_2$  for which the usual uncoupling is justified, and finally (v) what are the general qualitative features of the solutions.

### III. RESULTS AND CONCLUSIONS

The exact solution to the coupled integral equations (16) is derived in the Appendix. The answers to several of our questions are immediately apparent. First, solutions do exist, so that use of unsubtracted dispersion relations seems perfectly satisfactory. Not only do they exist, but they exist for all values of the constants  $\lambda_{ij}$ . Second, in regard to uniqueness, we find in the solutions we have obtained two arbitrary constants, which are determined by the two conditions that both the pion and nucleon form factors equal the electric charge at  $x=0$ . Now, we have assumed in our construction of these solutions that the form factors are both

integrable functions of  $x$ . An analysis of uncoupled equations of this type by Omnès<sup>4</sup> shows that in the uncoupled case arbitrarily strong poles may be inserted at threshold, and the solutions are therefore not unique. Thus it could be that if we relax our integrability condition, more solutions might be possible, but we do not know if this is actually the case for our coupled problem.

It makes no difference for our purpose whether further solutions can or cannot be constructed by this prescription. We only require that on physical grounds the actual physical form factors be everywhere finite. We must emphasize that this does not mean that the solutions of our model cannot have singularities, since we have replaced the physical kernels in Eq. (5) which vanish at the thresholds by constants which do not. It is only necessary to be sure that the solutions we use are those which would reduce to finite physical ones if the correct physical kernels had been used.

In our coupled problem we are unable to solve the equations with the physical kernels; therefore we are forced to guess which solutions of the coupled problem correspond to finite solutions of the physical problem. For the case that the eigenvalues of the matrix  $\lambda_{ij}$  are both positive, it is fairly easy to see that our integrable solutions are in the desired ones. To see this it is best to refer back to the uncoupled case. There the "pole-less" solution is

$$F(x) \sim (x - x_1)^{-(1/\pi) \tan^{-1} \lambda}. \quad (15)$$

If  $\lambda$  is positive this solution vanishes at infinity, is integrable, and goes over into a finite solution if the correct kernel is inserted. For the coupled case, when the eigenvalues of  $(\lambda_{ij})$  are positive, the analogous situation occurs; we find solutions containing the forms

$$(x - x_1)^{-(1/\pi) \tan^{-1} \lambda_i}, \quad (16)$$

where  $\lambda_i$  are the eigenvalues of  $(\lambda_{ij})$ . By analogy with the uncoupled case, we infer that these solutions correspond to "pole-less" solutions of this physical problem. However, if  $\lambda$  is negative, the solution (15) does not vanish as  $x \rightarrow \infty$ , but the solution obtained by inserting one pole, namely,

$$F(x) \sim (x - x_1)^{-(1/\pi) \tan^{-1} \lambda - 1} \quad (17)$$

does, and has an integrable singularity at  $x = x_1$ . In this case if the physical kernel is put in, the solution becomes one with a real pole at  $x_1$ ; thus (17) is not an admissible solution, even though it is integrable and vanishes at infinity.

In the coupled problem we face the analogous situation where one of the eigenvalues become negative. In that case a term of type (17) appears in the solution which we reject in the belief that it will become a pole in the "physical" problem just as in the uncoupled case. However, an additional solution to (14) also

<sup>4</sup> R. Omnès, Nuovo cimento 8, 316 (1958).

appears, which allows us sufficient freedom to satisfy all the conditions uniquely, even after we have thrown out terms of the type (17). But there seems to be no unambiguous method of determining whether this additional solution corresponds to a finite solution of the more physical problem or not. Because of this ambiguity we have concentrated our numerical evaluations on the positive case alone.

The remaining question is the validity of the standard uncoupling approximation, which neglects all but the lightest intermediate state. If  $x_2 > x_1$ , then this approximate solution is obtained by throwing out the state with the threshold at  $x_2$ ; thus Eqs. (14) are approximated by

$$\begin{aligned} F_1(x) &= \frac{\lambda_{11}}{\pi} \int_{x_1}^{\infty} \frac{F_1(x')}{x' - x} dx', \\ F_2(x) &= \frac{\lambda_{21}}{\pi} \int_{x_1}^{\infty} \frac{F_1(x')}{x' - x} dx'. \end{aligned} \quad (18)$$

The first of these is an integral equation for  $F_1(x)$ , the solution of which is:

$$\begin{aligned} F_1(x) &= \left( \frac{1}{(x/x_1) - 1} \right)^{\alpha_{11}} \frac{1}{(1 + \lambda_{11}^2)^{1/2}}, \quad x > x_1, \\ F_1(x) &= \left( \frac{1}{1 - (x/x_1)} \right)^{\alpha_{11}}, \quad x < x_1, \end{aligned} \quad (19)$$

where

$$\alpha_{11} = (1/\pi) \tan^{-1} \lambda_{11},$$

and the condition that  $F_1(0) = 1$  has been imposed.

To complete this approximate solution of Eqs. (14), we need  $F_2$ , which is clearly given from the second of Eqs. (18) as

$$F_2(x) = (\lambda_{21}/\lambda_{11}) F_1(x). \quad (20)$$

Now, however, a difficulty appears, because unless  $\lambda_{21} = \lambda_{11}$  it is impossible to satisfy the condition

$$F_2(0) = 1. \quad (21)$$

Thus, except for the trivial case  $\lambda_{11} = \lambda_{21}$ , it is impossible

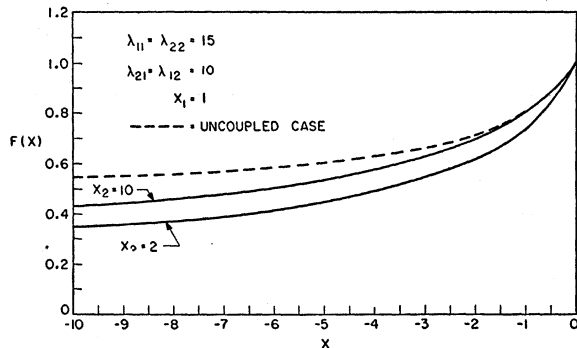


FIG. 3.  $F_2(x)$  for  $x < 0$  when  $x_1 = 1$ ,  $x_2 = 2, 10$  and infinity for  $\lambda_{11} = \lambda_{22} = 15$ ,  $\lambda_{12} = \lambda_{21} = 10$ .

to find solutions with the required properties from the approximate Eqs. (18). In order to obtain solutions in this case, it is necessary to abandon the no subtraction philosophy. The approximate Eqs. (18) should therefore be replaced by subtracted equations, namely:

$$\begin{aligned} F_1(x) &= 1 + \frac{\lambda_{11}}{\pi} x \int_{x_1}^{\infty} \frac{F_1(x')}{x'(x' - x)} dx', \\ F_2(x) &= 1 + \frac{\lambda_{21}}{\pi} x \int_{x_2}^{\infty} \frac{F_1(x')}{x'(x' - x)} dx'. \end{aligned} \quad (22)$$

The solution of these subtracted equations is still simple: for  $F_1$  we have the same result as before, but for  $F_2$  we now get

$$F_2(x) = 1 + (\lambda_{21}/\lambda_{11}) [F_1(x) - 1]. \quad (23)$$

The retention of the heavy intermediate state, with threshold at  $x_2$ , then, produces a significant difference in the high-energy behavior of the form factors, in that

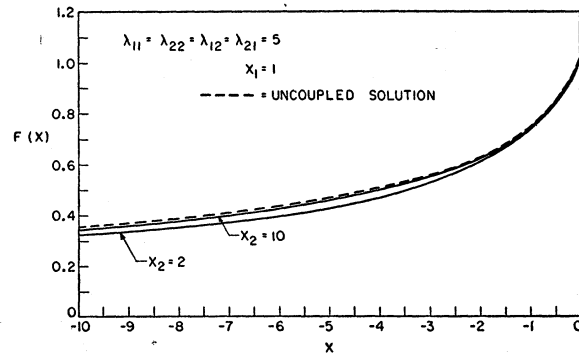


FIG. 4.  $F_2(x)$  for  $x < 0$  when  $x_1 = 1$ ,  $x_2 = 2, 10$  and infinity for  $\lambda_{11} = \lambda_{22} = \lambda_{12} = \lambda_{21} = 5$ .

if the heavy state is neglected, we have

$$\lim_{x \rightarrow \infty} F_2(x) = (1 - \lambda_{21}/\lambda_{11}), \quad (24)$$

whereas we have, of course,

$$\lim_{x \rightarrow \infty} F_2(x) = 0 \quad (25)$$

if the heavy state is included. It is interesting to observe that keeping the high mass state depresses the form factor  $F_2$  at high energies, instead of increasing it.

While the inclusion of both intermediate states makes a significant difference in the high-energy behavior, and thus determines the type of dispersion relation one may use, the numerical effects, at reasonable energies, of the heavy state are not noticeable unless the masses  $x_1$  and  $x_2$  get quite close.

Figures 3 through 6 show  $F_2(x)$  for negative values of  $x$ , for two values of  $x_2/x_1$ , namely 10 and 2, and also show  $F_2(x)$  on the basis of the uncoupling approximation. The overall shape is seen to be qualitatively insensitive to  $x_2/x_1$ ; in fact, the difference between the

value  $x_2/x_1=2$  and  $x_2/x_1=\infty$  (the uncoupled approximation) is not more than 30 or 40% even at the larger values of  $x$  plotted. This not only confirms the usual idea that if the heavy mass  $x_2$  is sufficiently large it has very little effect upon the form factors at low energy, but indicates that even when  $x_2/x_1=2$  the higher mass state does not produce qualitative differences even for  $x$  as large as  $-10$ . Figure 7 gives an example of what happens for  $x>0$ . This region is of somewhat less interest, since the model problem we have adopted has singularities at  $x=x_1$  and  $x=x_2$ , and these singularities presumably would not appear in any

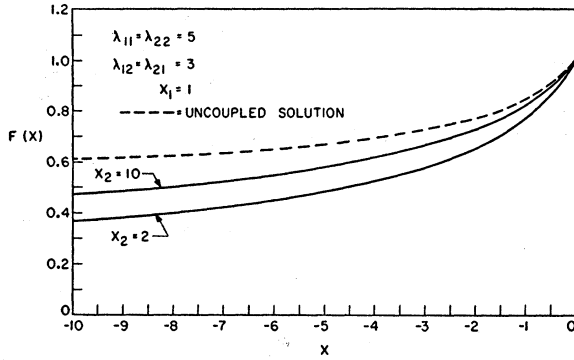


FIG. 5.  $F_2(x)$  for  $x<0$  when  $x_1=1$ ,  $x_2=2, 10$  and infinity for  $\lambda_{11}=\lambda_{22}=5$ ,  $\lambda_{12}=\lambda_{21}=3$ .

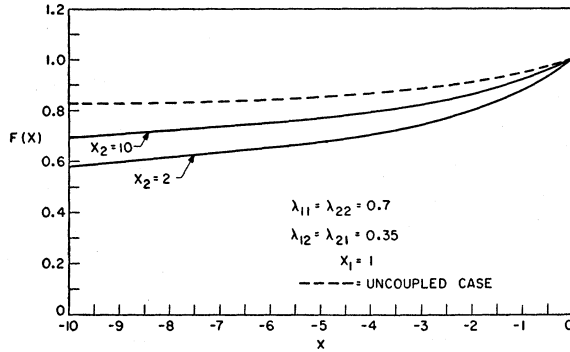


FIG. 6.  $F_2(x)$  for  $x<0$  when  $x_1=1$ ,  $x_2=2, 10$  and infinity for  $\lambda_{11}=\lambda_{22}=0.7$ ,  $\lambda_{12}=\lambda_{21}=0.35$ .

actual physical situation. The uncoupled approximate solution has a singularity only at  $x=x_1$ ; the great difference between the exact solutions and the uncoupled as in Fig. 7 is due just to this. For, as the pole is left behind, it is clear that the difference between the various cases are decreasing, so that at very large values of  $x$  the situation at  $x>0$  presumably will be similar to that for  $x<0$ .

To summarize, inclusion of a higher mass state permits the use of unsubtracted dispersion relations for both form factors. Unique solutions exist which go to zero at infinity and take on the value  $e$  at the origin, for all values of the coupling constants, while in the uncoupled case such unsubtracted dispersion relations

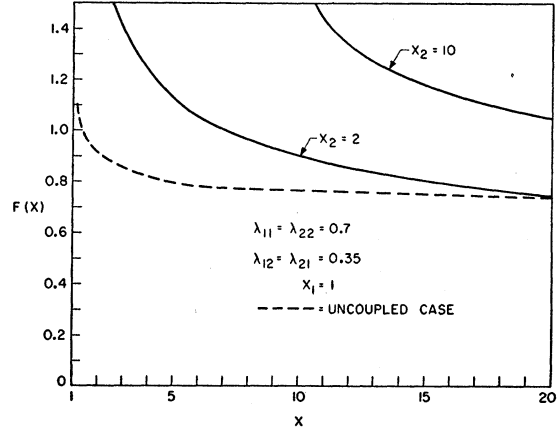


FIG. 7.  $F_2(x)$  for  $x>0$  when the same conditions as in Fig. 6.

are inconsistent with the necessary behavior of the form factors at the origin. However the form factors in the coupled case do not qualitatively differ from those of the uncoupled situation even when the mass ratio is not much greater than 1. This seems to bear out the validity of the uncoupling approximation for a wide range of  $x$ . Of course it would be unwarranted to assume that the neglect of higher mass states in the actual physical problem produces such a mild effect on the predicted form factors as in our simple model.

#### IV. ACKNOWLEDGMENTS

We are indebted to Professor Gordon Latta for suggesting the method by which the integral equations were solved, and for extensive discussions and other suggestions relative to the equations. We should also like to thank Allan Tucker for carrying out the numerical evaluations of the final results.

#### APPENDIX

It is convenient to make a scale change  $x \rightarrow (x-x_1)/(x_2-x_1)$  in equations so that the lower limits  $x_2$  and  $x_1$  are replaced by 1 and 0, and the conditions at  $x=0$  on  $F$  and  $G$  become  $F[-x_1/(x_2-x_1)] = G[-x_1/(x_2-x_1)] = 1$ .

We therefore consider the equations

$$\begin{aligned} F(x) &= \frac{\lambda_{11}}{\pi} P \int_1^\infty \frac{F(x')}{x'-x} dx' + \frac{\lambda_{12}}{\pi} P \int_0^\infty \frac{G(x')}{x'-x} dx', \\ G(x) &= \frac{\lambda_{21}}{\pi} P \int_1^\infty \frac{F(x')}{x'-x} dx' + \frac{\lambda_{22}}{\pi} P \int_0^\infty \frac{G(x')}{x'-x} dx', \end{aligned} \quad (1)$$

where  $\lambda_{12}=\lambda_{21}$ , and look for solutions which  $\rightarrow 0$  as  $x \rightarrow \infty$  and are locally integrable.

Take the Fourier transform of (1). The right-hand side of (1) can be written as a convolution by suitably extending the functions  $F(x)$  and  $G(x)$ , e.g., define

$$\begin{aligned} F_1(x) &= F(x) \quad x > 1 \\ &= 0 \quad x < 1, \end{aligned}$$

in which case

$$\int_1^\infty \frac{F(x')}{x'-x} dx' = \int_{-\infty}^\infty \frac{F_1(x')}{x'-x} dx'$$

Then, if we define

$$\begin{aligned} F_+(\omega) &= \int_1^\infty e^{i\omega x} F(x) dx, & G_+(\omega) &= \int_0^\infty e^{i\omega x} G(x) dx, \\ F_-(\omega) &= \int_{-\infty}^1 e^{i\omega x} F(x) dx, & G_-(\omega) &= \int_{-\infty}^0 e^{i\omega x} G(x) dx, \end{aligned} \quad (2)$$

and use the fact that

$$\int_{-\infty}^\infty \frac{e^{i\omega x}}{x} dx = i\pi\epsilon(\omega),$$

where

$$\epsilon(\omega) = \begin{cases} 1 & \omega > 0 \\ -1 & \omega < 0 \end{cases}$$

the Fourier Transform of (1) becomes

$$\begin{aligned} F_+(\omega) + F_-(\omega) &= -i\epsilon(\omega)[\lambda_{11}F_+(\omega) + \lambda_{12}G_+(\omega)], \\ G_+(\omega) + G_-(\omega) &= -i\epsilon(\omega)[\lambda_{21}F_+(\omega) + \lambda_{22}G_+(\omega)]. \end{aligned} \quad (3)$$

This is all for real  $\omega$ . The integrals (2) can be used to extend  $F_+$ ,  $F_-$ ,  $G_+$ ,  $G_-$  into the complex  $\omega$  plane. Let  $\omega = u + iv$ . Then (2) defines

$$\begin{aligned} F_+, G_+ & \text{ analytic for } v \geq 0, \\ F_-, G_- & \text{ analytic for } v \leq 0. \end{aligned}$$

Also as  $v \rightarrow \infty$ ,

$$F_+(\omega) \rightarrow O(e^{-|v|}), \quad G_+(\omega) \rightarrow O(1),$$

and as  $v \rightarrow -\infty$ ,

$$F_-(\omega) \rightarrow O(e^{|v|}), \quad G_-(\omega) \rightarrow O(1).$$

Finally, as  $u \rightarrow \pm\infty$ ,

$$F_\pm, G_\pm \rightarrow 0;$$

in fact, if  $F$  is bounded,  $F_{+,-}(\omega) \rightarrow 1/u$  as  $u \rightarrow \infty$ , while, if  $F$  has a singularity of type  $x^{-\alpha}$ ,  $\alpha < 1$ , then  $F_{+,-}(\omega) \rightarrow 1/u^{1-\alpha}$  as  $u \rightarrow \infty$ .

Let us introduce vector notation:

$$\mathfrak{F}_+ = \begin{pmatrix} F_+ \\ G_+ \end{pmatrix}, \quad \mathfrak{F}_- = \begin{pmatrix} F_- \\ G_- \end{pmatrix}, \quad \lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}. \quad (4)$$

Then (3) becomes

$$\mathfrak{F}_+(\omega) + \mathfrak{F}_-(\omega) = -i\epsilon(\omega)\lambda\mathfrak{F}_+(\omega). \quad (5)$$

We now want to decouple the equations (5). Let us choose  $S$  to diagonalize  $\lambda$ . Then

$$S\lambda S^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix};$$

i.e.,  $\lambda_1$  and  $\lambda_2$  are the eigenvalues of  $\lambda$ . If we define the functions

$$H_\pm = \sum_j S_{1j}(\mathfrak{F}_\pm)_j, \quad W_\pm = \sum_j S_{2j}(\mathfrak{F}_\pm)_j,$$

then Eq. (5) becomes two independent equations for  $H_\pm$  and  $W_\pm$ , namely:

$$[1 + i\epsilon(\omega)\lambda_1]H_+(\omega) = -H_-(\omega), \quad (6)$$

$$[1 + i\epsilon(\omega)\lambda_2]W_+(\omega) = -W_-(\omega). \quad (7)$$

For definiteness we shall assume the eigenvalues of  $\lambda$  satisfy  $\lambda_2 > \lambda_1 > 0$ .

We now have 2 independent equations, each with two unknowns. We would like to use these equations, coupled with known analytic properties of  $H_+$  and  $H_-$  when extended in complex plane, to determine  $H_+$  and  $H_-$ , and similarly  $W_+$  and  $W_-$ .

Let us first look at Eq. (6). Now  $H_+$  in this equation is the limiting value of a function analytic in the upper half plane as we approach real axis from above, since it is a linear combination of  $F_+$  and  $G_+$ . Likewise,  $H_-$  is the limiting value of a function analytic in lower half, and Eq. (6) is then a relation which holds for the limiting values of these analytic functions on the common boundary of their regions of analyticity.

We will now use Eq. (6) to characterize  $H_+$  everywhere. By means of analytic continuation, we will determine all the singular points of  $H_+(\omega)$ .

Suppose imaginary  $\omega = v > 0$ . Then  $H_+(\omega)$  is an analytic function. When we arrive at the negative real axis,  $H_+$  takes on value

$$H_+(\omega) = -H_-(\omega)/(1 - i\lambda_1) \quad (\omega < 0), \quad (8)$$

but  $-H_-/(1 - i\lambda_1)$  is an analytic function in the lower half plane. Thus on the right-hand side of Eq. (8), we have a function analytic in the lower half plane, and on the left-hand side of Eq. (8), we have a function analytic in the upper half plane. They are equal on their common boundary, the negative real axis. Therefore,  $-H_-/(1 - i\lambda_1)$  is the unique analytic extension of  $H_+$  into the lower half plane across the negative real axis. Thus, by using Eq. (8) as the definition of  $H_+$  in the lower half plane, we have been able to extend  $H_+$  to a single valued function in the entire plane excluding the positive real axis. For if we start off with  $H_+$  in the upper half plane or the lower half plane as defined by Eq. (8) and we go around any closed circuit (a) which does not cross the positive real axis, we come back to the same value of  $H_+$  (see Fig. 8).

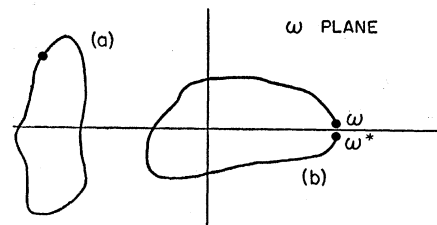


FIG. 8. Contours in  $\omega$  plane referred to in text.

However, consider a point  $\omega$  lying just above the positive real axis. Suppose we move around the contour (b) to a point  $\omega^*$  just below the real axis. We can use Eq. (6) for  $\omega > 0$  to relate  $H_+(\omega)$  and  $H_+(\omega^*)$ . This will give us the multivalued properties of  $H_+(\omega)$  across the positive real axis.

We have, in fact,

$$(1+i\lambda_1)H_+(\omega) = -H_-(\omega), \quad \omega > 0. \quad (9)$$

Therefore

$$H_+(\omega^*) = -H_-(\omega^*)/(1-i\lambda_1) \\ = [(1+i\lambda_1)/(1-i\lambda_1)]H_+(\omega) \quad (10)$$

as  $\text{Im}\omega \rightarrow 0$ .

Thus the analytic function defined by Eq. (8) is single valued in the cut plane and gets multiplied by a factor  $(1+i\lambda_1/1-i\lambda_1)$  when crossing the positive real axis. We look for an elementary function which has this property.

Consider  $\omega^{\alpha_1}$ , where  $\alpha_1 = (1/\pi) \tan^{-1}\lambda_1$ . Then

$$\omega^{\alpha_1} = \omega^{\alpha_1} \exp(2i \tan^{-1}\lambda_1) = \omega^{\alpha_1} [(1+i\lambda_1)/(1-i\lambda_1)].$$

(Note that  $0 \leq \alpha_1 \leq \frac{1}{2}$ ). Thus if we let

$$H_+(\omega) = \omega^{\alpha_1-1} \Phi(\omega),$$

then  $\Phi(\omega^*) = \Phi(\omega)$  and  $\Phi(\omega)$  is single valued in the entire complex  $\omega$  plane.

How about the local singularities of  $\Phi(\omega)$ ? If  $\Phi(\omega)$  has a pole at  $\omega \neq 0$  then so would  $H_+$ , but clearly from the integral representation of  $F_{\pm}$  and  $G_{\pm}$ ,  $H_+(\omega)$  is bounded at any point  $\omega \neq 0$ . At  $\omega = 0$ , clearly  $H_+(\omega)$  must be integrable for the inversion formula to hold; and hence must behave as  $\omega^{\gamma}$ ,  $\gamma > -1$ . This also follows from the fact  $F(x), G(x) \rightarrow 0$  as  $x \rightarrow \infty$ . [Note if  $F(x) \rightarrow 0$  faster than  $1/x$  then  $F_+(\omega)$  is bounded at  $\omega = 0$ .] Hence,  $\Phi(\omega) \sim \omega^{-\alpha_1+\gamma+1}$  near  $\omega = 0$  and hence  $\Phi(\omega)$  cannot have a pole at  $\omega = 0$ . Therefore  $\Phi(\omega)$  is an entire function. What about the behavior near  $\omega = \infty$  along lines parallel to the real axis? By the Reimann-Lebesgue lemma,

$$F_+(\omega), G_+(\omega) \rightarrow 0 \quad \text{as} \quad \text{Re}\omega \rightarrow \infty$$

(due to the rapidly oscillating exponential) and, indeed, if  $F(x)$  and  $G(x)$  are bounded,  $F_+(\omega), G_+(\omega) \sim 1/\omega$  near  $\text{Re}\omega = \infty$ . Hence  $\Phi(\omega)/\omega = H_+(\omega)\omega^{-\alpha_1} \rightarrow 0$  as  $\text{Re}\omega \rightarrow \infty$ . Thus we have the result that  $\Phi(\omega) \equiv [\Phi(\omega) - \Phi(0)]/\omega$  is an entire function which  $\rightarrow 0$  along any line parallel to the real axis. Now as  $\text{Im}\omega = v \rightarrow +\infty$ , from (2),  $H_+(\omega) \sim O(1)$  and hence  $\Phi(\omega) \sim O(1)$ ; similarly, as  $v \rightarrow -\infty$ ,  $\Phi(\omega) \rightarrow O(e^{|v|})$ . To summarize, then,  $\Phi(\omega)$  is an entire function which has the properties:

$$\begin{aligned} \Phi &\rightarrow 0 & \text{as } u &\rightarrow \pm\infty, \\ \Phi &\rightarrow O(1) & \text{as } v &\rightarrow +\infty, \\ \Phi &\rightarrow O(e^{|v|}) & \text{as } v &\rightarrow -\infty. \end{aligned} \quad (9)$$

It can be shown that a general representation for an entire function with properties (9) is

$$\Phi(\omega) = \int_0^1 e^{i\omega t} \varphi(t) dt, \quad (10)$$

where  $\varphi(t)$  is an arbitrary integrable function. Clearly any function of the form (10) is an entire function satisfying conditions (9). That any entire function which satisfied (9) can be written in form (10) is best seen by using Laplace's representation of an entire function.<sup>5</sup> Finally, then, we get the result

$$H_+(\omega) = \omega^{\alpha_1-1} \Phi + \omega^{\alpha_1} \int_0^1 e^{i\omega t} \varphi(t) dt, \quad (11)$$

where  $\Phi \equiv \Phi(0)$  is an arbitrary constant and  $\varphi(t)$  is an as yet undetermined integrable function.

Since Eqs. (6) and (7) are the same except for the replacement of  $\lambda_1$  by  $\lambda_2$ , and since  $W_+$  satisfies the same boundary conditions which were needed to obtain Eq. (11) for  $H_+$ , we immediately infer the corresponding representation for  $W_+$  to be

$$W_+(\omega) = \omega^{\alpha_2-1} \Psi + \omega^{\alpha_2} \int_0^1 e^{i\omega t} \psi(t) dt, \quad (12)$$

where  $\alpha_2 = (1/\pi) \tan^{-1}\lambda_2$  and  $\psi(t)$  is an arbitrary integrable function. Now

$$\begin{pmatrix} F_{\pm} \\ G_{\pm} \end{pmatrix} = S^{-1} \begin{pmatrix} H_{\pm} \\ W_{\pm} \end{pmatrix} = R \begin{pmatrix} H_{\pm} \\ W_{\pm} \end{pmatrix},$$

where  $R = S^{-1}$ . Let

$$R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}.$$

We can thus express  $F_+$  and  $G_+$  in terms of  $\psi(t)$  and  $\varphi(t)$  as follows: For  $\text{Im}\omega = v > 0$ :

$$F_+(\omega) = R_{11} \left[ \omega^{\alpha_1-1} \Phi + \omega^{\alpha_1} \int_0^1 \varphi(t) e^{i\omega t} dt \right] \\ + R_{12} \left[ \omega^{\alpha_2-1} \Psi + \omega^{\alpha_2} \int_0^1 \psi(t) e^{i\omega t} dt \right], \quad (13)(a)$$

$$G_+(\omega) = R_{21} \left[ \omega^{\alpha_1-1} \Phi + \omega^{\alpha_1} \int_0^1 \varphi(t) e^{i\omega t} dt \right] \\ + R_{22} \left[ \omega^{\alpha_2-1} \Psi + \omega^{\alpha_2} \int_0^1 \psi(t) e^{i\omega t} dt \right]. \quad (13)(b)$$

Now, for  $v < 0$ ,

$$\begin{aligned} H_-(\omega) &= -(1-i\lambda_1)H_+(\omega), \\ W_-(\omega) &= -(1-i\lambda_2)W_+(\omega). \end{aligned}$$

Thus

$$F_-(\omega) = -(1-i\lambda_1)R_{11} \left[ \omega^{\alpha_1-1} \Phi + \omega^{\alpha_1} \int_0^1 \varphi(t) e^{i\omega t} dt \right] \\ - (1-i\lambda_2)R_{12} \left[ \omega^{\alpha_2-1} \Psi + \omega^{\alpha_2} \int_0^1 \psi(t) e^{i\omega t} dt \right], \quad (13)(c)$$

<sup>5</sup> E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Clarendon Press, Oxford, 1937), p. 48.



and

$$G_-(\omega) = -(1-i\lambda_1)R_{21}\left[\omega^{\alpha_1-1}\Phi + \omega^{\alpha_1}\int_0^1 \varphi(t)e^{i\omega t}dt\right] \\ - (1-i\lambda_2)R_{22}\left[\omega^{\alpha_2-1}\Psi + \omega^{\alpha_2}\int_0^1 \psi(t)e^{i\omega t}dt\right]. \quad (13)(d)$$

For any functions  $\varphi(t), \psi(t)$  the transform equations (3) are satisfied and also  $F_+, G_+, F_-, G_- \rightarrow 0$  as  $u \rightarrow \pm \infty$ .

Clearly,  $G_+ = O(1)$  as  $v \rightarrow +\infty$  and  $F_- = O(e^{|\psi|})$  as  $v \rightarrow -\infty$ ; indeed these conditions were used to construct  $H_+$  and  $W_-$ . However, the stronger conditions

$$F_+ = O(e^{-|\psi|}) \quad \text{as } v \rightarrow +\infty,$$

and

$$G_- = O(1) \quad \text{as } v \rightarrow -\infty$$

have not been used and as is easily seen for an arbitrary (random)  $\varphi(t), \psi(t)$ , the construction of Eq. (13) gives

$$F_+ \sim O(1)$$

$$G_- \sim O(e^{|\psi|}).$$

It is just these stronger conditions that allow us to uniquely determine  $\varphi(t), \psi(t)$ , and we are left with a solution containing two unknown constants  $\Phi, \Psi$  which are then determined from the conditions

$$F(-x_1/(x_2-x_1)) = G(-x_1/(x_2-x_1)) = 1.$$

As indicated above, we need to look at the equations for  $F_+$  and  $G_-$ . The requirement that  $F_+(\omega) \sim O(e^{-|\psi|})$  means that the inverse Fourier transform of  $F_+(\omega)$ , that is  $F(x)$ , has no Fourier components less than 1; of course, that is how we obtained this condition. As a result, we must require that the inverse Fourier transform of (13)(a) vanish for  $x < 1$ . (It automatically vanishes for  $x < 0$ .) It is slightly more convenient to consider the function

$$\bar{F}_+(\omega) = [F_+(\omega)/R_{11}]\omega^{-\alpha_2} \\ = \omega^{-\alpha_2+\alpha_1-1}\Phi + \omega^{\alpha_1-\alpha_2}\int_0^1 \varphi(t)e^{i\omega t}dt \\ + \frac{R_{12}}{R_{11}}\left(\frac{\Psi}{\omega} + \int_0^1 \psi(t)e^{i\omega t}dt\right). \quad (14)$$

Now, just as explained above, for  $F_+(\omega)$ , the inverse Fourier transform of  $\bar{F}_+(\omega)$  must also vanish for  $0 < x < 1$ . Because the inversion is done in the upper half  $\omega$  plane  $\text{Im}\omega > 0$ , the inverse transform of  $1/\omega^{1+\gamma}$  exists and equals

$$\frac{x^\gamma}{i^{1+\gamma}\Gamma(1+\gamma)} \quad x > 0 \\ 0 \quad x < 0.$$

Using this result and the rule that the inverse Fourier transformation of a product of Fourier transforms is

the convolution of their individual Fourier transforms, we get

$$0 = \frac{\Phi x^\gamma}{i^{1+\gamma}\Gamma(1+\gamma)} + \frac{1}{i^\gamma\Gamma(\gamma)}\int_0^x \varphi(t)(x-t)^{\gamma-1}dt \\ + (R_{12}/R_{11})[-i\Psi + \psi(x)], \quad 0 < x < 1, \quad (15)$$

where  $\gamma = \alpha_2 - \alpha_1$ . Likewise, if we invert

$$\bar{G}_- = -\frac{G_-\omega^{-\alpha_2}}{R_{21}(1-i\lambda_1)} = \omega^{-1-\gamma}\Phi + \omega^{-\gamma}\int_0^1 \varphi(t)e^{i\omega t}dt \\ + \frac{R_{22}}{R_{21}}\left(\frac{1-i\lambda_2}{1-i\lambda_1}\right)\left[\frac{\Psi}{\omega} + \int_0^1 dt e^{i\omega t}\psi(t)\right],$$

we get (remember  $\text{Im}\omega < 0$  here)

$$0 = \frac{1}{(-i)^\gamma\Gamma(\gamma)}\int_x^1 dt \varphi(t)(t-x)^{\gamma-1} \\ + \frac{R_{22}}{R_{21}}\left(\frac{1-i\lambda_2}{1-i\lambda_1}\right)\psi(x), \quad (16)$$

where we have used the fact that the inverse Fourier transform of  $\omega^{-\gamma}$  ( $\text{Im}\omega < 0$ ) = 0 for  $x > 0$ .

Eliminating  $\psi(x)$  from Eqs. (15) and (16), we obtain

$$-i\Psi + \frac{R_{11}}{R_{12}}\frac{\Phi x^\gamma}{i^{1+\gamma}\Gamma(1+\gamma)} \\ = -\frac{R_{11}}{R_{12}}\frac{1}{i^\gamma\Gamma(\gamma)}\left[\int_0^x \varphi(t)(x-t)^{\gamma-1}dt \right. \\ \left. - \frac{1}{A}\int_x^1 \varphi(t)(t-x)^{\gamma-1}dt\right], \quad (17)$$

where

$$\frac{1}{A} = \frac{R_{21}R_{12}}{R_{22}R_{11}}\left(\frac{1-i\lambda_1}{1-i\lambda_2}\right)\frac{i^\gamma}{(-i)^\gamma} = \frac{R_{21}R_{12}}{R_{22}R_{11}}\left(\frac{1+\lambda_1^2}{1+\lambda_2^2}\right)^{\frac{1}{2}} < 0.$$

If we shift the range of integration in Eq. (17) to  $0 \rightarrow \infty$ , then transform techniques can also be applied to that equation. This is done by letting

$$t = \sigma/(1+\sigma), \quad x = \tau/(1+\tau).$$

Equation (17) then becomes

$$\int_0^\tau \varphi_1(\sigma)(\tau-\sigma)^{\gamma-1}d\sigma - \frac{1}{A}\int_\tau^\infty \varphi_1(\sigma)(\sigma-\tau)^{\gamma-1}d\sigma \\ = i\Psi\frac{R_{12}}{R_{11}}i^\gamma\Gamma(\gamma)(1+\tau)^{\gamma-1} - \frac{\Phi}{i^\gamma}\frac{\tau^\gamma}{1+\tau}, \quad (18)$$

where

$$\varphi_1(\sigma) \equiv \varphi(\sigma/(1+\sigma))[1/(1+\sigma)]^{1+\gamma}.$$

Equation (18) is a linear inhomogeneous integral

equation. If  $\varphi_{1\Psi}(\sigma)$  is a solution if only the  $\Psi$  term appeared in the inhomogeneous term, and  $\varphi_{1\Phi}(\sigma)$  a solution if the  $\Phi$  term appeared alone, then the general solution is  $\varphi_1(\sigma) = \varphi_{1\Phi}(\sigma) + \varphi_{1\Psi}(\sigma) + \text{arbitrary solution of the homogeneous equation}$ .

We then first calculate  $\varphi_{1\Psi}(\sigma)$ . In the process of doing this, we will also see there are no solutions to the homogeneous equation. Let  $\sigma = \tau u$  in Eq. (18), omitting the  $\Phi$  term. Then

$$\int_0^1 \varphi_{1\Psi}(\tau u)(1-u)^{\gamma-1} du - \frac{1}{A} \int_1^\infty \varphi_{1\Psi}(\tau u)(u-1)^{\gamma-1} du = i\Psi(R_{12}/R_{11})i^\gamma\Gamma(\gamma)\tau^{-\gamma}(1+\tau)^{\gamma-1} \equiv h(\tau). \quad (19)$$

Now, if we take the Mellin transform of Eq. (19), it becomes an algebraic equation. Explicitly, define:

$$\begin{aligned} \Phi_{1\Psi}(s) &= \int_0^1 u^{s-1} \varphi_{1\Psi}(u) du, \\ G(s) &= \int_0^1 u^{s-1}(1-u)^{\gamma-1} du - \frac{1}{A} \int_0^\infty du u^{s-1}(u-1)^{\gamma-1} \\ &= \frac{\Gamma(s)\Gamma(\gamma)}{\Gamma(s+\gamma)} - \frac{1}{A} \frac{\Gamma(1-s-\gamma)\Gamma(\gamma)}{\Gamma(1-s)}, \quad 1-\gamma > \text{Res} > 0, \\ H(s) &= i\Psi \frac{R_{12}}{R_{11}} i^\gamma \Gamma(\gamma) \frac{\Gamma(s-\gamma)\Gamma(1-s)}{\Gamma(1-\gamma)}, \quad \gamma < \text{Res} < 1. \end{aligned}$$

The Mellin transform  $F(s)$  of  $f(x)$  is defined as

$$F(s) = \int_0^\infty dx x^{s-1} f(x);$$

then if

$$f(x) = \int_0^\infty a(t)b(xt)db, \quad F(s) = A(1-s)B(s).$$

Thus Eq. (19) becomes

$$\Phi_{1\Psi}(s)G(1-s) = H(s), \quad (20)$$

where  $H(s)$  and  $G(1-s)$  exist only for  $\gamma < \text{Res} < 1$ . Now

$$G(1-s) = \frac{\pi\Gamma(\gamma)[\sin\pi(s-\gamma) - (1/A)\sin\pi s]}{\Gamma(s)\Gamma(1-s+\gamma)\sin\pi s\sin\pi(s-\gamma)}.$$

Also

$$\sin\pi(s-\gamma) - (1/A)\sin\pi s = D\sin\pi(s-\delta'),$$

where

$$\tan\pi\delta' = \frac{\sin\pi\gamma}{\cos\pi\gamma - 1/A} < \tan\pi\gamma,$$

and

$$D = |e^{-i\pi\gamma} - 1/A|.$$

For

$$\gamma < \text{Res} < 1, \quad 0 < \gamma - \delta' < \text{Re}(s - \delta') < 1 - \delta' < 1.$$

Thus  $G(1-s)$  never vanishes in the region  $\gamma < \text{Res} < 1$

and there are no solutions to the homogeneous equation

$$\Phi_{1\Psi=0}(s)G(1-s) = 0,$$

except

$$\Phi_{1\Psi=0}(s) = 0.$$

For  $\Psi \neq 0$  we get the solution

$$\Phi_{1\Psi}(s) = \frac{\pi i^{\gamma+1} R_{12} \csc\pi(s-\delta')}{D R_{11} \Gamma(1-\gamma)}. \quad (21)$$

Inverting Eq. (21) produces

$$\varphi_{1\Psi}(\tau) = \frac{\Psi i^{\gamma+1} R_{12} \tau^{-\delta'}}{\Gamma(1-\gamma) D R_{11} (1+\tau)},$$

or

$$\varphi_{1\Psi}(t) = i^{\gamma+1} \frac{R_{12}}{R_{11}} \frac{\Psi t^{-\delta'}}{\Gamma(1-\gamma)} \frac{(1-t)^{\delta'-\gamma}}{D}. \quad (22)$$

Finally, using Eq. (15) to obtain  $\psi_{\Psi}(x)$ , we get

$$\psi_{\Psi}(x) = \frac{i\Psi \sin\pi\gamma}{\pi A D} \frac{d}{dx} \int_x^1 dt t^{-\delta'} (1-t)^{\delta'-\gamma} (t-x)^{\gamma}. \quad (23)$$

We can obtain  $\varphi_{\Phi}(t)$  and  $\psi_{\Phi}(t)$  by noting that our original expressions in Eq. (13) for  $F_+$  and  $F_-$ ,  $G_+$  and  $G_-$  are left unchanged by the interchange  $\Phi \leftrightarrow \Psi$ ,  $\varphi(x) \leftrightarrow \psi(x)$ ,  $R_{11} \leftrightarrow R_{12}$ ,  $R_{21} \leftrightarrow R_{22}$ ,  $\lambda_1 \leftrightarrow \lambda_2$ ,  $\alpha_1 \leftrightarrow \alpha_2$ . Under such a substitution  $\gamma \leftrightarrow -\gamma$  and  $A \leftrightarrow 1/A$  and  $\delta' \leftrightarrow -\delta$ , where

$$\tan\pi\delta = \sin\pi\gamma / (\cos\pi\gamma + A).$$

Thus we get

$$\psi_{\Phi}(t) = i^{1-\gamma} \frac{R_{11}}{R_{12}} \frac{\Phi t^{\delta} (1-t)^{\gamma-\delta}}{R_{12} \Gamma(1+\gamma) |e^{i\pi\gamma} - A|}, \quad (24)$$

and

$$\varphi_{\Phi}(x) = -i \frac{\Phi \sin\pi\gamma}{\pi(1/A) |e^{i\pi\gamma} - A|} \frac{d}{dx}$$

$$\times \int_x^1 dt t^{\delta} (1-t)^{\gamma-\delta} (t-x)^{-\gamma}. \quad (25)$$

Now

$$\varphi(x) = \varphi_{\Phi}(x) + \varphi_{\Psi}(x),$$

and

$$\psi(x) = \psi_{\Phi}(x) + \psi_{\Psi}(x).$$

If we insert the expressions from Eqs. (21) through (25) into Eq. (13), we get  $F_+(\omega)$ ,  $F_-(\omega)$ ,  $G_+(\omega)$  and  $G_-(\omega)$ . Inverting these Fourier transforms, we get the final formulas for  $F(x)$  and  $G(x)$ , as shown below. The arbitrary constants  $\Phi$ ,  $\Psi$  are then determined by setting

$$F(-x_1/(x_2-x_1)) = G(-x_1/(x_2-x_1)) = 1.$$

The final results are listed below. Only the solutions for  $F(x)$  are explicitly given; the solutions for  $G(x)$  are easily obtained from the ones for  $F(x)$  by replacing  $R_{11}$  by  $R_{21}$ ,  $R_{12}$  by  $R_{22}$ , and by multiplying the two single

integral terms by  $(R_{12}R_{21}/R_{11}R_{22})$ . In addition, in going from  $F$  to  $G$  it is necessary to note the  $F$  solution for  $x > 1$  corresponds to the  $G$  solution for  $x > 0$ , except that if  $x < 1$  in the  $G$  solutions, the upper limits on the

integrations over  $dy$  should be  $x$  instead of 1. Similarly, the  $F$  solution for  $x < 1$  corresponds to the  $G$  solution for  $x < 0$ ; consequently this  $G$  solution has none of the modifications that  $F$  has in the range  $0 < x < 1$ .

### Solutions

(i) For  $x > 1$ ,

$$F(x) = R_{11}\Phi \left\{ x^{-\alpha_1} - \frac{\sin\pi\gamma}{\pi\gamma} \left| \frac{1}{e^{i\pi\gamma} - A} \right| \left[ A \frac{d}{dx} \int_0^1 dy (x-y)^{-\alpha_1} \frac{d}{dy} \int_y^1 dt (t-y)^{-\gamma} t^\delta (1-t)^{\gamma-\delta} \right. \right. \\ \left. \left. + \frac{\Gamma(1-\gamma)\Gamma(1-\alpha_1)}{\Gamma(1-\alpha_2)} \frac{d}{dx} \int_0^1 dy (x-y)^{-\alpha_2} y^\delta (1-y)^{\gamma-\delta} \right] \right\} + R_{12}\Psi \left\{ x^{-\alpha_2} - \frac{\sin\pi\gamma}{\pi\gamma} \left| \frac{1}{e^{i\pi\gamma} - 1/A} \right| \right. \\ \left. \times \left[ \frac{1}{A} \frac{d}{dx} \int_0^1 dy (x-y)^{-\alpha_2} \frac{d}{dy} \int_y^1 dt (t-y)^{\gamma} t^{-\delta'} (1-t)^{\delta'-\gamma} \right. \right. \\ \left. \left. + \frac{\Gamma(1+\gamma)\Gamma(1-\alpha_2)}{\Gamma(1-\alpha_1)} \frac{d}{dx} \int_0^1 dy (x-y)^{-\alpha_1} y^{-\delta'} (1-y)^{\delta'-\gamma} \right] \right\}.$$

(ii) For  $x < 1$

$$F(x) = R_{11}(1+\lambda_1^2)^{\frac{1}{2}}\Phi \left\{ (-x)^{-\alpha_1} - \frac{\sin\pi\gamma}{\pi\gamma} \left| \frac{1}{e^{i\pi\gamma} - A} \right| \left[ A \frac{d}{dx} \int_x^1 dy (y-x)^{-\alpha_1} \frac{d}{dy} \int_y^1 dt (t-y)^{-\gamma} t^\delta (1-t)^{\gamma-\delta} \right. \right. \\ \left. \left. + \left( \frac{1+\lambda_2^2}{1+\lambda_1^2} \right)^{\frac{1}{2}} \frac{\Gamma(1-\gamma)\Gamma(1-\alpha_1)}{\Gamma(1-\alpha_2)} \frac{d}{dx} \int_x^1 dy (y-x)^{-\alpha_2} y^\delta (1-y)^{\gamma-\delta} \right] \right\} \\ + R_{12}(1+\lambda_2^2)^{\frac{1}{2}}\Psi \left\{ (-x)^{-\alpha_2} - \frac{\sin\pi\gamma}{\pi\gamma} \left| \frac{1}{e^{i\pi\gamma} - 1/A} \right| \left[ \frac{1}{A} \frac{d}{dx} \int_x^1 dy (y-x)^{-\alpha_2} \frac{d}{dy} \int_y^1 dt (t-y)^{\gamma} t^{-\delta'} (1-t)^{\delta'-\gamma} \right. \right. \\ \left. \left. + \left( \frac{1+\lambda_1^2}{1+\lambda_2^2} \right)^{\frac{1}{2}} \frac{\Gamma(1+\gamma)\Gamma(1-\alpha_2)}{\Gamma(1-\alpha_1)} \frac{d}{dx} \int_x^1 dy (y-x)^{-\alpha_1} y^{-\delta'} (1-y)^{\delta'-\gamma} \right] \right\}.$$

Note: If  $0 < x < 1$  drop terms  $(-x)^{-\alpha_1}$  and  $(-x)^{-\alpha_2}$ ; if  $x < 0$  make lower limits on integrals over  $y$  zero instead of  $x$ .

This discussion has all referred to the situation where  $\lambda_2 > \lambda_1 > 0$ . When one of the eigenvalues is zero; i.e., when  $\lambda_{11}\lambda_{22} = \lambda_{12}^2$ , then the equations can be uncoupled. The equation for  $F_1(x)$  is a simple Omnés type equation which is easily solved, and  $F_2(x) = (\lambda_{12}/\lambda_{11})F_1(x)$ . In this case there is no solution which satisfies  $F_1(0) = F_2(0) = 1$  unless  $\lambda_{12} = \lambda_{11}$ , much like the situation where the two nucleon state was neglected.

When both eigenvalues  $\lambda_2$  and  $\lambda_1$  are less than zero the solution contains terms of the form  $\Phi x^{-\alpha_1-1}$  and  $\Psi x^{-\alpha_2-1}$  which we anticipate would become poles in the physical problem. This requires we set  $\Phi = \Psi = 0$ . This

gives us a null solution since the remaining terms are all proportional to  $\Phi$  or  $\Psi$ .

If  $\lambda_2 > 0$  and  $\lambda_1 < 0$ , the same general method is applicable and the solution which again is quite involved contains terms of the form  $\Phi x^{-\alpha_1-1}$  and  $\Psi x^{-\alpha_2}$ . Based upon an analogy with the uncoupled case we then reject the terms proportional to  $\Phi$ . However, for this case the homogeneous integral equation analogous to (18)(a) possesses a solution and so we are still left with two arbitrary parameters. Then we could satisfy the conditions  $F(0) = G(0) = 1$ , if we accepted this solution. However, since it is now impossible to compare with the uncoupled case, we have no unambiguous way of determining whether this solution will give rise to a pole in the physical problem. It is for this reason we have omitted detailed discussion of this case.